JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 139, 243–267 (1989)

On Domains of Some Nonlinear Operators

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Submitted by Ky Fan

Received June 11, 1987

Let $T: \mathscr{E} \to H$ be a nonlinear, disjointly additive operator defined on a space of simple functions $\mathscr{E}$, H being an arbitrary F-space. The following problem is considered: to extend the domain of T and find a maximal function F-space E with the Lebesgue property such that $T: E \to H$ is continuous. Next, the problem of continuity and equicontinuity of a sequence of such operators is discussed. © 1989 Academic Press, Inc.

1. INTRODUCTION

In this paper the following problem is considered: given a disjointly additive operator T which maps simple functions into an F-space H, we want to extend the domain of the definition of T to a broader subset of the space M of all measurable functions and construct a maximal F-function space $E \subset M$ such that $T: E \to H$ is a continuous operator. We will solve this natural and essential problem using some means of the theory of modular function spaces which seems to be the appropriate tool. The construction presented in this paper is applicable primarily to nonlinear operators. Nevertheless, sometimes we can determine domains of nonlinear operators by means of extended domains of some special linear integral operators (cf. Theorem 3.12 and Example 3.15). For the details of the theory of extended domains of linear integral transformations the reader is referred to [1, 10, 14, 15]. The general method of construction described in the following paper follows some suggestions contained in [6, Example 8.5]. These suggestions were based on the theory of nonlinear operator measures (cf. [2, 4, 8, 9, 13]). In the following paper we will construct the

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0022-247X/89 $3.00

domain of T directly, without referring to the theory of nonlinear operator valued measures. In the last part of the paper the problem of continuity and equicontinuity of a sequence of such operators is discussed.

2. Preliminaries

Let X be a nonempty set and $\mathscr{P}$ be a nontrivial δ-ring of subsets of X. Let Σ be the smallest σ-algebra of subsets of X such that Σ contains $\mathscr{P}$ and

(i) $\mathscr{P}$ is an ideal in Σ, i.e., $E \cap A \in \mathscr{P}$ for every $E \in \mathscr{P}$, $A \in \Sigma$,

(ii) there exists a nondecreasing sequence of sets $X_i \in \mathscr{P}$, such that $X = \bigcup_{i=1}^{\infty} X_i$.

By $\mathscr{E}$ we shall denote the linear space of all simple real valued functions of the form

$$g = \sum_{i=1}^{n} r_i 1_{A_i}, \qquad \text{where} \quad r_i \in \mathbb{R},\ A_i \in \mathscr{P},\ A_i \cap A_j = \emptyset \quad \text{for} \quad i \neq j.$$

The symbol 1_A stands for the characteristic function of the set A. By M we shall denote the space of all measurable functions, i.e., all functions $f: X \to \mathbb{R}$ such that there exists a sequence $g_n \in \mathscr{E}$, $|g_n| \leqslant |f|$, and $g_n(x) \to f(x)$ for every $x \in X$.

Definition 2.1. A set function $\mu: \Sigma \to [0, \infty]$ will be called a σ-subadditive measure if and only if:

1. $\mu(\emptyset) = 0$,
2. $\mu(\bigcup_{n=1}^{\infty} A_n) \leqslant \sum_{n=1}^{\infty} \mu(A_n)$ for each sequence of sets (A_n) from Σ,
3. $\mu(A) \leqslant \mu(B)$ if $A, B \in \Sigma$, $A \subset B$.

Definition 2.2. (cf. [5–7]). A functional $\rho: M \times \Sigma \to [0, \infty]$ is called a function semimodular if and only if the following conditions are satisfied:

(A1) $\rho(0, A) = 0$ for each $A \in \Sigma$;

(A2) If $A \in \Sigma$, f and $g \in M$, and $|f(x)| \leqslant |g(x)|$ for all $x \in A$ then $\rho(f, A) \leqslant \rho(g, A)$;

(A3) For every $f \in M$, $\rho(f, \cdot): \Sigma \to [0, \infty]$ is a σ-subadditive measure;

(A4) For every $A \in \mathscr{P}$, $\rho(\alpha, A) \to 0$ as $\alpha \to 0^+$, where for the sake of simplicity we wrote $\rho(\alpha, A)$ instead of $\rho(\alpha 1_A, A)$;

(A5) There exists $\alpha_0 > 0$ such that, if $\rho(\alpha, A) = 0$ for an $\alpha > \alpha_0$ $(A \in \mathscr{P})$, then $\rho(\beta, A) = 0$ for every $\beta > 0$;

(A6) $\rho(\alpha, \cdot)$ is order continuous on $\mathscr{P}$ for all $\alpha > 0$, i.e., $\rho(\alpha, A_n) \to 0$ as $A_n \downarrow \emptyset$, $A_n \in \mathscr{P}$;

(A7) $\rho(f, A) = \sup\{\rho(g, A);\ g \in \mathscr{E},\ |g(x)| \leqslant |f(x)| \text{ for every } x \in A\}$.

For the sake of simplicity we will write $\rho(f)$ in place of $\rho(f, X)$.

Typical examples of function modulars are

(a) $\rho(f, A) = \int_A \phi(x, f(x))\, dm(x)$ (Musielak–Orlicz modular),

(b) $\rho(f, A) = \sup_{m \in \mathscr{M}} \int_A \phi(x, f(x))\, dm(x)$,

where $\mathscr{M}$ denotes a family of nonnegative measures and $\phi: X \times \mathbb{R} \to \mathbb{R}_+$ satisfies (see [3])

(1) $\phi(x, u)$ is a nondecreasing, continuous, even function of u such that $\phi(x, 0) = 0$, $\phi(x, u) > 0$ for $u \neq 0$, $\phi(x, u) \to \infty$ as $u \to \infty$,

(2) $\phi(x, u)$ is a measurable function of x for all $u \in \mathbb{R}$.

In the sequel such functions will be called ϕ-functions.

Let us give another example which will be discussed later in this paper:

(c) $$\rho(f, A) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\rho_n(f, A)}{1 + \rho_n(f, A)},$$

where

$$\rho(f, A) = \sup_{m \in \mathscr{M}} \int_A \phi_n(x, f(x))\, dm(x),$$

ϕ_n being ϕ-functions for $n \in \mathbb{N}$.

Let $\mathscr{X}$ be an arbitrary vector space. Let us recall that a functional $\rho: \mathscr{X} \to [0, \infty]$ is called a semimodular if:

(1) $\rho(0) = 0$ and $f = 0$ if $\rho(\lambda f) = 0$ for every $\lambda > 0$,

(2) $\rho(\alpha f) = \rho(f)$ for $|\alpha| = 1$,

(3) $\rho(\alpha f + \beta g) \leqslant \rho(f) + \rho(g)$ if $\alpha + \beta = 1$ and $\alpha, \beta \geqslant 0$.

If instead of (1) there holds

(1′) $\rho(f) = 0$ if and only if $f = 0$,

then ρ is called a modular. Clearly, every modular is a semimodular. A semimodular ρ defines the modular space, i.e., the vector space $\mathscr{X}_\rho$ given by

$$\mathscr{X}_\rho = \{f \in \mathscr{X};\ \rho(\lambda f) \to 0 \text{ as } \lambda \to 0^+\}.$$

$\mathscr{X}_\rho$ can be equipped with the F-norm (so called Luxemburg's F-norm) defined by

$$\|f\|_\rho = \inf\{\alpha > 0;\ \rho(f/\alpha) \leqslant \alpha\}; \qquad \text{see [11, 12].}$$

Let us recall that a functional $\|\cdot\|: \mathscr{X} \to \mathbb{R}^+$ is called an F-norm if and only if:

(1) $\|f\|=0$ iff $f=0$;

(2) $\|\alpha f\| = \|f\|$ if α is a scalar and $|\alpha|=1$;

(3) $\|f+g\| \leqslant \|f\| + \|g\|$;

(4) $\|\alpha_k f_k - \alpha f\| \to 0$ if $\alpha_k \to \alpha$ and $\|f-f_k\| \to 0$.

The F-norm induces a metric $d(f, g)=\|f-g\|$; the topology given by this metric is linear, i.e., both vector space operations are continuous. If this metric is complete then the linear topological space obtained in this way is called an F-space.

It is a basic fact of the modular space theory that the F-norm convergence $\|f_n\|_\rho \to 0$ is equivalent to $\rho(\alpha f_n) \to 0$ for every $\alpha > 0$.

If a modular ρ is defined by the formula

$$\rho(f) = \int_X \phi(x, f(x))\, dm(x), \qquad \phi \text{ being a } \phi\text{-function},$$

then the corresponding modular space is denoted by L^ϕ and is usually called the Musielak–Orlicz space (Orlicz space if the function ϕ does not depend directly on the first variable). For the review of the theory of modular and Orlicz spaces we refer the reader to the book of Musielak [12].

PROPOSITION 2.3 [7, Theorem 2.1.4]. *After identifying functions which differ only on ρ-null sets (i.e., sets $A \in \Sigma$ such that $\rho(\alpha, A)=0$ for all $\alpha>0$) the functional $\rho(\cdot, X)$: $M \to [0, \infty]$ is a semimodular.*

If α_0 from (A5) is equal to zero then ρ is a modular and will be called a function modular. Let ρ be a function semimodular. According to the general modular theory we can define now a modular space

$$L_\rho = \{f \in M; \rho(\lambda f) \to 0 \text{ as } \lambda \to 0^+\},$$

which is called a modular function space. For further detail the reader is referred to [5–7]. The following properties of modular function spaces were proved there.

PROPOSITION 2.4. 1. *$(L_\rho, \|\cdot\|_\rho)$ is a complete space and the F-norm $\|\cdot\|_\rho$ is monotone with respect to the natural order in M.*

2. *If there is a number $\lambda>0$ such that $\rho(\lambda(f_n - f)) \to 0$ then there exists a subsequence (g_n) such that $g_n \to f$ ρ–a.e.*

3. *If $f_n(x) \to f(x)$ ρ–a.e. in X then there exists a nondecreasing sequence of sets $H_k \in \mathscr{P}$ such that $H_k \uparrow X$ and f_n converges uniformly to f on every set H_k.*

4. *$L_\rho \supset L_\rho^0 \supset E_\rho$, where*

$$E_\rho = \{f \in L_\rho;\ \lambda f \in L_\rho^0 \text{ for every } \lambda > 0\}$$

and

$$L_\rho^0 = \{f \in M;\ \rho(f, \cdot\,) \text{ is order continuous, i.e., } \rho(f, D_k) \downarrow 0 \text{ whenever } D_k \downarrow 0\}.$$

5. *E_ρ has the Lebesgue property, i.e., $\|f 1_{D_k}\|_\rho \to 0$ if $f \in E_\rho$ and $D_k \downarrow 0$.*

6. *E_ρ is the closure of $\mathscr{E}$ (in the sense of the F-norm $\|\cdot\|_\rho$).*

7. *The uniform convergence on sets from $\mathscr{P}$ is stronger then the $\|\cdot\|_\rho$-convergence.*

DEFINITION 2.5. An F-space $E \subset M$ will be called a function F-space, whenever the F-norm $\|\cdot\|_E$ is monotone with respect to the natural order in M and

$$E = \{f \in M;\ \|\lambda f\|_E \to 0 \text{ as } \lambda \to 0\}.$$

For a given nonlinear operator $T: \mathscr{E} \to H$, where H is an arbitrary F-space, we may define the quantities

$$\omega_\delta(T, \alpha, A) = \sup\{\|T(f) - T(g)\|_H; f, g \in \mathscr{E}, \operatorname{supp}(f) \subset A, \operatorname{supp}(g) \subset A,$$
$$|f(x)| \leqslant \alpha, |g(x)| \leqslant \alpha, |f(x) - g(x)| \leqslant \delta \text{ for all } x \in X\},$$

where $A \in \mathscr{P}$ and $\operatorname{supp}(f) = \{x \in X; f(x) \neq 0\}$.

3. RESULTS

Let H be an F-space. Throughout this paper we will assume that the operator $T: \mathscr{E} \to H$ satisfies the following conditions:

T is additive, i.e., $T(f+g) = T(f) + (g)$ for every $f, g \in \mathscr{E}$ such that $\operatorname{supp}(f) \cap \operatorname{supp}(g) = \varnothing$; (3.1)

For every $\alpha > 0$ and $A \in \mathscr{P}$, $\omega_\delta(T, \alpha, A) \to 0$ as $\delta \to 0$; (3.2)

For every $\alpha > 0$ and $A_k \in \mathscr{P}$ with $A_k \downarrow \varnothing$,
$\sup\{\|T(g)\|_H; g \in \mathscr{E}, |g(x)| \leqslant \alpha 1_{A_k}(x)\} \to 0$ as $k \to \infty$; (3.3)

There exists a number $\alpha_0 > 0$ such that $T(f) = 0$ for every $f \in \mathscr{E}$ if for a certain $\alpha > \alpha_0$, $\sup\{\|T(g)\|_H; g \in \mathscr{E}, |g(x)| \leqslant \alpha$ for $x \in X\} = 0$. (3.4)

DEFINITION 3.5. For every $f \in M$ and $A \in \Sigma$ we set

$$\rho(f, A) = \sup\{\|T(g)\|_H; g \in \mathscr{E}, |g(x)| \leqslant 1_A(x)|f(x)| \text{ for ll } x \in X\}.$$

THEOREM 3.6. *The functional* $\rho: M \times \Sigma \to [0, \infty]$ *is a function semimodular.*

Proof. We have to prove that ρ has the properties (A1) through (A7) from Definition 2.2. Since T is additive then $T(0) = 0$ and therefore (A1) holds. The monotonicity of ρ is an immediate consequence of the following inclusion which holds for $A \in \Sigma$, $f, g \in M$, $|f(x)| \leqslant |g(x)|$ for all $x \in A$:

$$\{\|T(h)\|_H; h \in \mathscr{E}, |h(x)| \leqslant 1_A(x)|f(x)| \text{ for } x \in X\}$$
$$\subset \{\|T(h)\|_H; h \in \mathscr{E}, |h(x)| \leqslant 1_A(x)|g(x)| \text{ for } x \in X\}.$$

In order to prove (A3) let us observe first that $\rho(f, \varnothing) = 0$ since $1_\varnothing = 0$. Moreover, if $A \subset B$ then $\rho(f, A) \leqslant \rho(f, B)$ because

$$\{\|T(h)\|_H; h \in \mathscr{E}, |h(x)| \leqslant 1_A(x)|f(x)| \text{ for all } x \in X\}$$
$$\subset \{\|T(h)\|_H; h \in \mathscr{E}, |h(x)| \leqslant 1_B(x)|f(x)| \text{ for all } x \in X\}.$$

To prove the σ-subadditivity of $\rho(f, \cdot)$ let us take first a sequence (A_n) of mutually disjoint sets from Σ and let us denote $A = \bigcup_{n=1}^\infty A_n$. Let us take a function $g \in \mathscr{E}$ such that $\operatorname{supp}(g) \subset A$ and compute

$$\|T(g)\|_H = \|T(g1_A) + T(g1_{A'})\|_H \leqslant \|T(g1_A)\|_H$$
$$\leqslant \sum_{i=1}^N \|T(g1_{A_n})\|_H + \|T(g1_{B_N})\|,$$

where $A' = X \backslash A$ and $B_N = \bigcup_{n=N+1}^\infty A_n$. Since $g \in \mathscr{E}$ and $B_N \downarrow \varnothing$ it follows from (3.3) that

$$\|T(g1_{B_N})\|_H \to 0 \quad \text{as} \quad N \to \infty.$$

Hence

$$\|T(g)\|_H \leqslant \sum_{i=1}^\infty \|T(g1_{A_n})\|_H$$

and

$$\begin{aligned}
\rho(f, A) &\\
&= \sup\{\|T(g)\|_H; g\in\mathscr{E}, |g(x)| \leqslant 1_A(x)|f(x)| \text{ for } x\in X\}\\
&\leqslant \sup\left\{\sum_{n=1}^{\infty} \|T(g1_{A_n})\|_H; g\in\mathscr{E}, |g(x)| \leqslant 1_A(x)|f(x)| \text{ for } x\in X\right\}\\
&\leqslant \sum_{n=1}^{\infty} \sup\{\|T(g1_{A_n})\|_H; g\in\mathscr{E}, |g(x)| \leqslant 1_A(x)\,|f(x)| \text{ for } x\in X\}\\
&\leqslant \sum_{n=1}^{\infty} \sup\{\|T(g1_{A_n})\|_H; g\in\mathscr{E}, 1_{A_n}(x)|g(x)| \leqslant 1_{A_n}(x)|f(x)| \text{ for } x\in X\}\\
&\leqslant \sum_{n=1}^{\infty} \sup\{\|T(h)\|_H; h\in\mathscr{E}, |h(x)| \leqslant 1_{A_n}(x)|f(x)| \text{ for } x\in X\}\\
&\leqslant \sum_{n=1}^{\infty} \rho(f, A_n).
\end{aligned}$$

Finally,

$$\rho\left(f, \bigcup_{n=1}^{\infty} A_n\right) \leqslant \sum_{n=1}^{\infty} \rho(f, A_n)$$

for disjointly (A_n). Since

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} C_n, \qquad \text{where} \quad C_n = A_n \Big\backslash \bigcup_{i=1}^{n-1} A_i,$$

then

$$\rho\left(f, \bigcup_{n=1}^{\infty} A_n\right) = \rho\left(f, \bigcup_{n=1}^{\infty} C_n\right) \leqslant \sum_{n=1}^{\infty} \rho(f, C_n) \leqslant \sum_{n=1}^{\infty} \rho(f, A_n).$$

The property (A4) follows from (3.2) via

$$\rho(\alpha, A) \leqslant \omega_\alpha(T, \alpha_0, A) \to 0 \qquad \text{for} \quad \alpha\to 0, \alpha\leqslant\alpha_0.$$

The property (A5) follows immediately from (3.4) while (A6) has been assumed in (3.3). It remains to prove (A7). Since $\rho(\cdot, A)$ is nondecreasing it follows that $\rho(g, A) \leqslant \rho(f, A)$ for every $g\in\mathscr{E}$ such that $|g(x)| \leqslant |f(x)|$ for every $x\in A$ and consequently

$$\sup\{\rho(g, A); g\in\mathscr{E}, |g(x)| \leqslant |f(x)| \text{ for every } x\in A\} \leqslant \rho(f, A).$$

On the other hand, for every $g \in \mathscr{E}$ with supp $g \subset A$

$$\|T(g)\|_H \leqslant \rho(g, A)$$

and finally

$$\begin{aligned}\rho(f, A) &= \sup\{\|T(g)\|_H; g \in \mathscr{E}, |g(x)| \leqslant 1_A(x)|f(x)| \text{ for all } x \in X\} \\ &\leqslant \sup\{\rho(g, A); g \in \mathscr{E}, |g(x)| \leqslant 1_A(x)|f(x)| \text{ for all } x \in X\} \\ &\leqslant \sup\{\rho(g, A); g \in \mathscr{E}, |g(x)| \leqslant |f(x)| \text{ for all } x \in A\}.\end{aligned}$$

The next theorem is the main result of the paper.

THEOREM 3.7. (i) *We can extend T to the set L_ρ^0 by the formula*

$$T(f) = \lim_{n \to \infty} T(f_n), \qquad \textit{where} \quad f_n \in \mathscr{E}, f_n \to f \quad \rho\text{-a.e.}$$

and

$$|f_n(x)| \leqslant |f(x)| \qquad \textit{for} \quad n \in \mathbb{N} \qquad \textit{and} \qquad x \in X.$$

(ii) *This definition does not depend on the choice of such a sequence of simple functions.*

(iii) *The operator $T\colon L_\rho^0 \to H$ is additive.*

(iv) *The inequality $\|T(f1_A)\|_H \leqslant \rho(f, A)$ holds for every $f \in L_\rho^0$, $A \in \Sigma$.*

(v) *$\|T(f1_{A_k})\|_H \to 0$ whenever $f \in L_\rho^0$, $A_k \downarrow \varnothing$.*

(vi) *$T\colon E_\rho \to H$ is a continuous operator.*

(vii) *E_ρ is the largest function F-space with the Lebesgue property such that T is continuous in E_ρ.*

Proof. Ad. (i). Take a function $f \in L_\rho^0$; since f is measurable it follows that there exists a sequence of simple functions (f_n) such that $f_n \to f$ ρ-a.e., $|f_n| \leqslant |f|$. By Proposition 2.4 we can pick up a sequence of sets $H_k \in \mathscr{P}$ such that $\bigcup_{k=1}^{\infty} H_k = X$ and f_n converges uniformly to the function f on every set H_k. Let us fix an arbitrary $\varepsilon > 0$. Since $f \in L_\rho^0$, there is a natural k such that $\rho(f, D_k) < \varepsilon/4$, where $D_k = X \backslash H_k$. We may find $G_k \subset H_k$ so that the function $f1_{G_k}$ is bounded and $\rho(f, H_k \backslash G_k) < \varepsilon/8$. Consequently the functions $f_m 1_{G_k}$, $f_n 1_{G_k}$ are bounded. Recall that the subadditive measure $\rho(f, \cdot)$ is order continuous because $f \in L_\rho^0$. Denote $Z_k = H_k \backslash G_k$ and compute

$$\begin{aligned}&\|T(f_n 1_{H_k}) - T(f_m 1_{H_k})\|_H \\ &\quad\leqslant \|T(f_n 1_{G_k}) - T(f_m 1_{G_k})\|_H + \|T(f_n 1_{Z_k})\|_H + \|T(f_m 1_{Z_k})\|_H \\ &\quad\leqslant \omega_{\delta_{n,m}}(T, \alpha, G_k) + 2\rho(f, Z_k) \leqslant \varepsilon/2\end{aligned}$$

for n, m sufficiently large, where $\alpha > 0$ was chosen in such a way that

$$\sup\{|f_n(x)|; x \in G_k\} \leqslant \sup\{|f(x)|; x \in G_k\} < \alpha \qquad \text{for} \quad n \geqslant N_0,$$

where we denoted

$$\delta_{n.m} = \sup\{|f_n(x)| - f_m(x)|; x \in G_k\} \to 0 \qquad \text{as} \quad n, m \to \infty.$$

We have

$$\begin{aligned} &\|T(f_n) - T(f_m)\|_H \\ &\quad \leqslant \|T(f_n 1_{H_k}) - T(f_m 1_{H_k})\|_H + \|T(f_n 1_{D_k})\|_H + \|T(f_m 1_{D_k})\|_H \\ &\quad \leqslant \|T(f_n 1_{H_k}) - T(f_m 1_{H_k})\|_H + 2\rho(f, D_k) < \varepsilon. \end{aligned}$$

We conclude that the limit $\lim_{n \to \infty} T(f_n)$ exists because H is complete.

Ad. (ii). The limit does not depend on the choice of the sequence (f_n). Indeed, take another such sequence $g_n \in \mathscr{E}$. We may take then the sequence $(f_1, g_1, f_2, g_2, ...)$ which satisfies the same conditions. Consequently, $\lim_{n \to \infty} T(f_n) = \lim_{n \to \infty} T(g_n)$.

Ad. (iii). This is evident in view of additivity of T on simple functions.

Ad. (iv). Given $f \in L^0_\rho$ and $A \in \Sigma$, we have

$$T(f 1_A) = \lim_{n \to \infty} T(s_n 1_A) \qquad \text{where} \quad s_n \in \mathscr{E},\ |s_n| \leqslant |f|, \quad \text{and} \quad s_n \to f \quad \rho\text{-a.e.}$$

Fix arbitrary $\varepsilon > 0$. For n sufficiently large there holds

$$\|T(f 1_A) - T(s_n 1_A)\|_H < \varepsilon.$$

Hence,

$$\|T(f 1_A)\|_H < \varepsilon + \|T(s_n 1_A)\|_H \leqslant \varepsilon + \rho(f, A),$$

which yields

$$\|T(f 1_A)\|_H \leqslant \rho(f, A).$$

Ad. (v). Immediate consequence of (iv).

Ad. (vi). In order to get continuity of $T: E_\rho \to H$ we will prove that every sequence (f_n) converging to f in E_ρ contains a subsequence (g_n) such that

$$\|T(g_n) - T(f)\|_H \to 0.$$

Assume, therefore, that $f_n \in E_\rho$ and $f \in E_\rho$ and $\|f_n - f\|_\rho \to 0$. By Proposition 2.4(2) there exists a subsequence (g_n) of (f_n) such that $g_n \to f$ ρ–a.e. By Proposition 2.4(3) we can select a sequence (H_k) such that $H_k \in \mathscr{P}$, $H_k \uparrow X$, and (g_n) converges uniformly to f on every H_k. Fix $\varepsilon > 0$; because $f \in E_\rho$, we may take $k \in \mathbb{N}$ such that

$$\rho(2f, W_k) < \varepsilon/8, \qquad \text{where} \quad W_k = X \backslash H_k$$

and, because $\rho(2(g_n - f)) \to 0$, $n_1 \in \mathbb{N}$ such that

$$\rho(2(g_n - f)) < \varepsilon/8 \qquad \text{for} \quad n \geqslant n_1.$$

Thus, for $n \geqslant n_1$

$$\rho(g_n, W_k) \leqslant \rho(2(g_n - f), W_k) + \rho(2f, W_k) < \varepsilon/4.$$

Let us consider the following inequalities $(n \geqslant n_1)$:

$$\begin{aligned} \|T(g_n) - T(f)\|_H &\leqslant \|T(g_n 1_{H_k}) - T(f 1_{H_k})\|_H + \|T(g_n 1_{W_k})\|_H + \|T(f 1_{W_k})U_H \\ &\leqslant \|T(g_n 1_{H_k}) - T(f 1_{H_k})\|_H + \rho(g_n, W_k) + \rho(f, W_k) \\ &\leqslant \|T(g_n 1_{H_k}) - T(f 1_{H_k})\|_H + \varepsilon/4 + \varepsilon/8 \\ &\leqslant \|T(g_n 1_{H_k}) - T(f 1_{H_k})\|_H + \varepsilon/2. \end{aligned}$$

Let (G_m) be a sequence of sets from $\mathscr{P}$ such that $G_m \uparrow H_k$ and $f 1_{G_m}$ are bounded for all m. Similarly as it was done above we can take m and $n_2 \geqslant n_1$ such that

$$\rho(2f, H_k \backslash G_m) < \varepsilon/4 \qquad \text{for} \quad n \geqslant n_2.$$

Since (g_n) uniformly converges to f on G_m, it follows by (3.2) that there exists n_3 such that

$$\|T(g_n 1_{G_m}) - T(f 1_{G_m})\|_H < \varepsilon \qquad \text{for} \quad n \geqslant n_3 \geqslant n_2 \geqslant n_1.$$

Hence,

$$\|T(g_n 1_{H_k}) - T(f 1_{H_k})\|_H \leqslant \varepsilon + \|T(g_n 1_{H_{k,m}})\|_H + \|T(f 1_{H_{k,m}})\|_H \leqslant \varepsilon + \varepsilon/2,$$

where $H_{k,m} = H_k \backslash G_m$. Finally, $\|T(g_n) - T(f)\|_H < 2\varepsilon$, which completes the proof of part (vi).

Ad. (vii). Assume $E \subset M$ is a function F-space with the Lebesgue property such that $T: E \to H$ is continuous but $E \backslash E_\rho \neq \varnothing$. Take a function $f \in E \backslash E_\rho$; there exists a sequence $D_k \downarrow \varnothing$ such that $\|f 1_{D_k}\|_\rho$ does not converge to zero, which gives (passing to a subsequence if necessary) positive constants γ and α such that

$$\rho(\alpha f 1_{D_k}) > \gamma > 0 \qquad \text{for all natural} \quad k.$$

There exist, therefore, functions $g_k \in \mathscr{E}$ such that

$$|g_k| \leqslant \alpha |f 1_{D_k}| \qquad \text{and} \qquad \|T(g_k)\|_H > \gamma/2.$$

By the monotonicity and the absolute continuity of the F-norm in E we may conclude, however, that

$$\|g_k\|_E \leqslant \|\alpha f 1_{D_k}\|_E \to 0.$$

In view of the continuity of $T: E \to H$ we get

$$0 < \gamma/2 \leqslant \|T(g_k)\|_E \to 0.$$

The contradiction completes the proof of (vii) and of the entire Theorem 3.7.

EXAMPLE 3.8. Let us consider the Nemytskii operator

$$T(f)(x) = \phi(x, f(x))$$

defined for every simple, Lebesgue measurable function $f: I \to \mathbb{R}$, where $I = [0, 1]$, ϕ is a ϕ-function (see Section 2), and $H = L^1(I, m)$, m denotes the Lebesgue measure in I. Orthogonal additivity of the operator T is obvious. Assume additionally that the function $\phi: I \times \mathbb{R} \to \mathbb{R}^+$ is continuous, the function $\phi(x, \cdot)$ is even for every $x \in I$, and the function $\phi(\cdot, \alpha)$ is summable for every positive α. Then, the condition (3.2) follows easily by the uniform continuity of ϕ on every set of the form $Ix[0, \alpha]$ while the summability of the function $\phi(\cdot, \alpha)$ gives immediately the property (3.3). The property (3.4) follows from the fact that $\phi(x, u) > 0$ for $u > 0$. Take any measurable $f: I \to \mathbb{R}$. We have

$$\rho(f) = \sup\left\{\int_I \phi(x, g(x))\, dm(x), |g| \leqslant |f|, g \in \mathscr{E}\right\} = \int_I \phi(x, f(x))\, dm(x),$$

which means simply that ρ is the Musielak–Orlicz modular. In view of Theorem 3.7 the Nemytskii operator T acts continuously from E_ρ to L^1. In this case E_ρ coincides with E^ϕ, the subspace of all finite elements of the Musielak–Orlicz space L^ϕ, defined by

$$E^\phi = \left\{f \in M; \int_I \phi(x, \lambda f(x))\, dm(x) < \infty \text{ for every } \lambda > 0\right\}.$$

EXAMPLE 3.9. Let ϕ be a ϕ-function which satisfies the following Lipschitz-type condition:

to every $\alpha > 0$ there exists a constant M_α such that

$$|\phi(x, u) - \phi(x, v)| \leqslant M_\alpha |u - v| \tag{3.9.a}$$

holds for every u and v with $|u| \leqslant \alpha$, $|v| \leqslant \alpha$ and for almost all x in $\mathbb{R}$.

Assume again that the function $\phi(\cdot, \alpha)$ is summable for every $\alpha > 0$. Let us assume that $k: \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ is a measurable function. By Σ we denote the σ-algebra of Lebesgue measurable subsets from $\mathbb{R}$ and $\mathscr{P}$ stands for the δ-ring of sets of finite measure. Assume that $k(x,y) > 0$ for all $x \in \mathbb{R}$ and $y \in Z$, where $Z \subset \mathbb{R}$ is of positive measure. Let us consider the Hammerstein operator

$$T(f)(x) = \int_{\mathbb{R}} k(x, y)\, \phi(y, f(y))\, dm(y), \qquad f \in \mathscr{E}.$$

The operator T is additive; (3.2) follows from (3.9.a). If $H = L^\infty(\mathbb{R})$, then

$$\rho(f) = \operatorname*{ess\,sup}_{x \in \mathbb{R}} \int_{\mathbb{R}} k(x, y)\, \phi(y, f(y))\, dm(y)$$

and

$$E_\rho = \left\{ f \in M; \forall \lambda > 0\ \forall D_k \downarrow \varnothing\ \operatorname*{ess\,sup}_{x \in \mathbb{R}} \int_{D_k} k(x, y)\, \phi(y, \lambda f(y))\, dm(y) \to 0 \right\}.$$

Suppose that the kernel is degenerate, i.e., the function k is bounded and does not depend on the second variable, then (3.3) easily follows from the summability of functions of the form $\phi(\cdot, \alpha)$. Property (3.4) holds in view of strict positivity of ϕ on $\mathbb{R} \times Z$. Moreover $E_\rho = E^\phi$.

Assume now that for a $p > 1$ there exists a constant M such that

$$\int_{\mathbb{R}} k(x, y)^p\, dm(y) < M < \infty \qquad \text{holds for all} \quad x \in \mathbb{R}.$$

In view of the Hölder inequality we get then that the condition (3.3) is satisfied and that $E_\rho \supset E^\psi$, where $\psi(x, u) = [\phi(x, u)]^q$, $p^{-1} + q^{-1} = 1$. Indeed, let $f \in E^\psi$ and $A_k \downarrow \varnothing$. Then

$$\begin{aligned} &\operatorname*{ess\,sup}_{x \in \mathbb{R}} \int_{A_k} k(x, y)\, \phi(y, f(y))\, dm(y) \\ &\quad \leqslant \operatorname*{ess\,sup}_{x \in \mathbb{R}} \left(\int_{A_k} [k(x, y)]^p\, dm(y) \right)^{1/p} \left(\int_{A_k} [\phi(y, f(y))]^q\, dm(y) \right)^{1/q} \\ &\quad \leqslant M^{1/p} \left(\int_{A_k} [\phi(y, f(y))]^q\, dm(y) \right)^{1/q} \to 0. \end{aligned}$$

The strict inclusion may hold here when the kernel is not degenerate. For instance, let $k(x, y) = 0$ for $y \geqslant 0$ and all $x \in \mathbb{R}$ while $k(x, y) > 0$ otherwise. If $D \subset \mathbb{R}^+$, $\lambda > 0$ then

$$\operatorname*{ess\,sup}_{x \in \mathbb{R}} \int_D k(x, y)\, \phi(y, \lambda f(y))\, dm(y) = 0$$

holds for every measurable function f. Hence, for every sequence of measurable sets $A_k \downarrow \varnothing$ there holds

$$\operatorname*{ess\,sup}_{x \in \mathbb{R}} \int_{A_k} k(x, y)\, \phi(y, \lambda f(y))\, dm(y) \to 0$$

for every function f such that $f 1_{\mathbb{R}}$ belongs to E^{ψ}, even when f itself is not a member of E^{ψ}. Let us also observe that the space E_{ρ} will be different from any Musielak–Orlicz space if we drop the positivity or monotonicity of the function ϕ.

EXAMPLE 3.10. Consider the operator

$$T(f)(x) = \int_{\mathbb{R}} e^{-itx}\, \phi(f(t))\, dm(t),$$

i.e., the nonlinear version of the Fourier transformation. The function $\phi\colon \mathbb{R} \to \mathbb{R}$ is an increasing, continuous, odd function such that

$$\lim_{u \to +\infty} \phi(u) = +\infty \qquad \text{and} \qquad \lim_{u \to -\infty} \phi(u) = -\infty.$$

Let $H = L^2(\mathbb{R})$ and let Σ, $\mathscr{P}$ be the same as in Example 3.9. We will prove that $E_{\rho} = E^{\psi}$, where ρ is a function modular induced by the nonlinear operator T and $\psi(u) = [\phi(u)]^2$.

Let $f \in E^{\psi}$. Since for every function $\alpha > 0$ and $g \in \mathscr{E}$ the function $\phi(\alpha g(\cdot))$ belongs to L^2 and the Fourier transformation $\mathscr{F}\colon L^2 \to L^2$ is an isometry (up to the constant $c = \sqrt{2\pi}$) then we get

$$\begin{aligned}
\rho(\alpha f, A_k) &= \sup\left\{\left(\int_{\mathbb{R}} \left|\int_{A_k} e^{-itx}\, \phi(\alpha g(t))\, dm(t)\right|^2 dm(x)\right)^{1/2}; g \in \mathscr{E}, |g| \leqslant |f|\right\}\\
&= \sup\left\{\left(\int_{\mathbb{R}} \left|\mathscr{F}(1_{A_k}(\cdot)\, \phi(\alpha g(\cdot)))(x)\right|^2 dm(x)\right)^{1/2}; g \in \mathscr{E}, |g| \leqslant |f|\right\}\\
&= \sup\{\|\mathscr{F}(1_{A_k}(\cdot)\, \phi(\alpha g(\cdot)))\|_{L^2}; g \in \mathscr{E}, |g| \leqslant |f|\}\\
&= c \sup\{\|1_{A_k}(\cdot)\, \phi(\alpha g(\cdot))\|_{L^2}; g \in \mathscr{E}, |g| \leqslant |f|\}\\
&= c \sup\left\{\left(\int_{A_k} \phi^2(\alpha g(t))\, dm(t)\right)^{1/2}; g \in \mathscr{E}, |g| \leqslant |f|\right\}\\
&= c\left(\int_{A_k} \phi^2(\alpha f(t))\, dm(t)\right)^{1/2} = c\left(\int_{A_k} \psi(\alpha f(t))\, dm(t)\right)^{1/2} \to 0.
\end{aligned}$$

Hence, $E^{\psi} \subset E_{\rho}$.

In order to prove the inverse inclusion let us fix an $\alpha > 0$ and a sequence $A_k \downarrow \varnothing$. Given $f \in E_\rho$. We have then: $\rho(\alpha f, A_k) \to 0$ as $k \to \infty$. For every simple function $g_k \in \mathscr{E}$ such that $|g_k| \leqslant |f 1_{A_k}|$ we get

$$\left(\int_{\mathbb{R}} \left|\int_{\mathbb{R}} e^{-itx} \phi(\alpha g_k(t))\, dm(t)\right|^2 dm(x)\right)^{1/2} \leqslant \rho(\alpha f, A_k) \to 0,$$

which means that $\mathscr{F}(\phi(g_k(\cdot))) \to 0$, where $\mathscr{F}$ denotes the Fourier transformation. Since $\mathscr{E} \subset E^\phi$, it follows that the function $\phi(\alpha g_k(\cdot))$ is a member of the space L^2. The inverse to the Fourier transformation acts continuously from L^2 onto L^2 and therefore the functions $\phi(\alpha g_k(\cdot))$ converge to zero in L^2, i.e.,

$$\int_{\mathbb{R}} \phi^2(\alpha g_k(t))\, dm(t) \to 0.$$

We claim now that $f \in E^\psi$, where $\psi(u) = (\phi(u))^2$. Indeed, assume to the contrary that

$$\int_{\mathbb{R}} \phi^2(\alpha f(t))\, dm(t) = \infty$$

for an $\alpha > 0$. There exist, therefore, a number $\varepsilon > 0$ and a sequence $A_k \downarrow \varnothing$ such that

$$\int_{\mathbb{R}} 1_{A_k}(t)\, \phi^2(\alpha f(t))\, dm(t) > \varepsilon.$$

Thus, we may choose a sequence of simple functions $g_k \in \mathscr{E}$ so that $|g_k| \leqslant |f 1_{A_k}|$ and

$$\int_{\mathbb{R}} \phi^2(\alpha g_k(t))\, dm(t) > \frac{\varepsilon}{2}.$$

We proved, however, that the integrals

$$\int_{\mathbb{R}} \phi^2(\alpha g_k(t))\, dm(t) \to 0 \qquad \text{as} \quad k \to \infty.$$

The contradiction completes the proof of the equality $E_\rho = E^\psi$.

EXAMPLE 3.11. Let X, Σ, $\mathscr{P}$, m be the same as in the previous two examples and let $k: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a measurable function such that $k(x, \cdot) \in L^1$ m–a.e. Let ϕ be a function with the same properties as in the previous example. Let us consider the Hammerstein operator of the form

$$T(f)(x) = \int_{\mathbb{R}} k(x, y)\, \phi(f(y))\, dm(y).$$

Let $H = M$ be the space of all measurable functions regarded as a topological vector space with the topology of convergence in measure on all subsets of finite measure. This topology may be defined by means of an F-norm

$$\|f\|_m = \int_{\mathbb{R}} \frac{|f(x)|}{1 + |f(x)|} p(x)\, dm(x),$$

where $p > 0$ m–a.e. is a measurable function such that

$$\int_{\mathbb{R}} p(x)\, dm(x) = 1.$$

Let us define a linear integral operator K by the formula

$$Kf(x) = \int_{\mathbb{R}} k(x, y) f(y)\, dm(y).$$

As the domain of this operator we can take the space

$$D_K = \left\{ f \in M; \int_{\mathbb{R}} |k(x, y)|\, |f(y)|\, dm(y) < \infty \right\}.$$

The set D_K is usually called the proper domain of the linear operator K. Assume additionally that K is nonsingular, i.e., there exists $g \in D_K$ such that $g > 0$ m–a.e.

For every $f \in M$ let us put

$$\|f\|_{\sim} = \|f\|_m + d_K(f),$$

where

$$d_K(f) = \sup\{ \|Kg\|_m ; g \in D_K, |g| \leqslant |f| \ m\text{–a.e.} \}.$$

It is well known from the theory of linear integral operators (cf. [10, 14]) that $\|\cdot\|_{\sim}$ is a complete translation invariant metric on an additive group M. Denoting by $\tilde{D}_K$ the closure in $(M, \|\cdot\|_{\sim})$ of D_K we get the extended domain of the operator K, i.e., K can be extended to a continuous operator $\tilde{K}\colon \tilde{D}_K \to M$ (cf., [10, 14]).

THEOREM 3.12. *The following two conditions are equivalent*:

(a) *$f \in E_\rho$, where ρ is a function semimodular given by the operator T*;

(b) *The function $\phi(\lambda f(\cdot) \in \tilde{D}_K$ for every positive λ.*

Proof. Let us observe that for every $f \in M$ and $A \in \Sigma$,

$$\rho(f, A) = \sup \left\{ \left\| \int_A k(\cdot, y)\, \phi(g(y))\, dm(y) \right\|_m ; \right.$$

$$\left. |g| \leqslant |f|\, 1_A \ m\text{–a.e.}, \phi(g(\cdot)) \in D_K \right\}. \tag{3.13}$$

Indeed, if g is a measurable, bounded function then

$$\int_{\mathbb{R}} |k(x, y)| \, |\phi(g(y))| \, dm(y) \leqslant \operatorname*{ess\,sup}_{z \in \mathbb{R}} |\phi(g(z))| \cdot \int_{\mathbb{R}} |k(x, y)| \, dm(y) < \infty$$

for almost all x because $k(x, \cdot) \in L^1(\mathbb{R})$ m–a.e. Therefore $\phi(g(\cdot)) \in D_K$. Thus,

$$\begin{aligned}\rho(f, A) &= \sup \left\{ \left\| \int_{\mathbb{R}} k(\cdot, y) \, \phi(g(y)) \, dm(y) \right\|_m ; g \in \mathscr{E}, |g| \leqslant |f| \, 1_A \right\} \\ &\leqslant \sup \left\{ \left\| \int_A k(\cdot, y) \, \phi(g(y)) \, dm(y) \right\|_m ; \phi(g(\cdot)) \in D_K, |g| \leqslant |f| \, 1_A \right\}.\end{aligned}$$

To prove the equality let us take a function g such that $|g| \leqslant |f|$ m–a.e. and $\phi(g(\cdot)) \in D_K$. There exists a sequence (g_k) of $\mathscr{P}$-simple functions such that $|g_k| \uparrow |g|$ m–a.e. Since for every $A \in \Sigma$ we have

$$\int_A |k(x, y)| \, |\phi(g(y))| \, dm(y) < \infty \qquad m\text{–a.e.},$$

it follows that

$$\begin{aligned}&\lim_{n \to \infty} \int_A k(x, y) \, \phi(g_n(y)) \, dm(y) \\ &\qquad = \int_A k(x, y) \, \phi(g(y)) \, dm(y) \qquad m\text{–a.e.}\end{aligned}$$

Thus

$$\left\| \int_A k(\cdot, y) \, \phi(g_n(y)) \, dm(y) - \int_A k(\cdot, y) \, \phi(g(y)) \, dm(y) \right\|_m \to 0$$

and consequently to every $\varepsilon > 0$ there corresponds $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}&\left\| \int_A k(\cdot, y) \, \phi(g(y)) \, dm(y) \right\|_m \\ &\qquad \leqslant \varepsilon + \left\| \int_A k(\cdot, y) \, \phi(g_{n_0}(y)) \, dm(y) \right\|_m .\end{aligned}$$

In view of arbitrariness of $\varepsilon > 0$ we get finally (3.13). Let the functions h and g be related by $h(y) = \phi(g(y))$. Since ϕ is an odd function we have $|h(y)| = \phi(|g(y)|)$. On the other hand, ϕ is increasing and therefore $|h(y)| \leqslant |\phi(f(y))| \, 1_A(y)$ if and only if $|g(y)| \leqslant |f(y)| \, 1_A(y)$. Finally,

$$\begin{aligned}&\rho(f, A) \\ &\quad = \sup \left\{ \left\| \int_A k(\cdot, y) \, h(y) \, dm(y) \right\|_m ; h \in D_K, |h(y)| \leqslant |\phi(f(y))| \, 1_A \ m\text{–a.e.} \right\} \\ &\quad = d_K(\phi(f(\cdot)) \, 1_A). \qquad (3.14)\end{aligned}$$

Assume now that (a) holds and a function $f \in E_\rho$. Let us define

$$Z_n = \{x \in [-n, n]; |f(x)| \leqslant n\} \qquad \text{and} \qquad A_n = X \setminus Z_n.$$

Fix arbitrary $\lambda > 0$. Since $\lambda f 1_{Z_n}$ is measurable and bounded for almost all $x \in \mathbb{R}$ then

$$\phi(\lambda f 1_{Z_n}(\cdot)) \in D_K.$$

We have

$$\begin{aligned} d_K(\phi(\lambda f(\cdot) - \phi(\lambda f 1_{Z_n}(\cdot))) &= d_K(\phi(\lambda f(\cdot)\, 1_{A_n}(\cdot))) \\ &= \rho(\lambda f, A_n) \to 0. \end{aligned}$$

Since

$$\|\phi(\lambda f(\cdot)) - \phi(\lambda f(\cdot)\, 1_{Z_n}(\cdot))\|_m \to 0$$

we get

$$\|\phi(\lambda f(\cdot)) - \phi(\lambda f(\cdot)\, 1_{Z_n}(\cdot))\|_{\sim} \to 0,$$

which means that

$$\phi(\lambda f(\cdot)) \in \tilde{D}_K.$$

To prove that (b) implies (a), suppose that f is such that $\phi(\lambda f(\cdot)) \in \tilde{D}_K$ for every $\lambda > 0$. Let us fix $\lambda > 0$ and $A_k \downarrow \varnothing$. Since $\phi(\lambda f(\cdot)) \in \tilde{D}_K$ there exists a sequence $g_n \in D_K$ such that $\|g_n - \phi(\lambda f(\cdot))\|_{\sim} \to 0$. In particular, $d_K(g_n - \phi(\lambda f(\cdot))) \to 0$. For a given $\varepsilon > 0$ let us fix $n \in \mathbb{N}$ such that

$$d_K(g_n - \phi(\lambda f(\cdot))) < \varepsilon/2.$$

Observe that

$$\begin{aligned} &d_K(1_{A_k} g_n) \\ &\quad = \sup\left\{\left\|\int_{A_k} k(\cdot, y)\, g(y)\, dm(y)\right\|_m ; g \in D_K, |g(y)| \leqslant |g_n(y)|\, 1_{A_k}(y)\ m\text{–a.e.}\right\} \\ &\quad \leqslant \sup\left\{\left\|\int_{A_k} |k(\cdot, y)|\, |g(y)|\, dm(y)\right\|_m ; g \in D_K, |g(y)| \right. \\ &\qquad\qquad\qquad\qquad \left. \leqslant |g_n(y)|\, 1_{A_k}(y)\ m\text{–a.e.}\right\} \\ &\quad \leqslant \left\|\int_{A_k} |k(\cdot, y)|\, |g(y)|\, dm(y)\right\|_m \to 0, \end{aligned}$$

because

$$\int_{\mathbb{R}} |k(\cdot, y)|\, |g(y)|\, dm(y) < \infty \qquad m\text{–a.e.}$$

By (3.14),

$$\begin{aligned}\rho(\lambda f, A_k) &= d_K(\phi(\lambda f(\cdot))\, 1_{A_k}) \\ &\leqslant d_K(g_n 1_{A_k} - \phi(\lambda f(\cdot)) 1_{A_k}) + d_K(g_n 1_{A_k}) \\ &\leqslant d_K(g_n - \phi(\lambda f(\cdot))) + d_K(g_n 1_{A_k}) < \varepsilon/2 + \varepsilon/2 = \varepsilon\end{aligned}$$

for k sufficiently large. This means that f belongs to E_ρ, which completes the proof.

Theorem 3.12 states that a function f belongs to E_ρ which is a domain of the nonlinear operator

$$T(f)(x) = \int_{\mathbb{R}} k(x, y)\, \phi(f(y))\, dm(y)$$

if and only if for every $\lambda > 0$ the function $y \mapsto \phi(\lambda f(y))$ belongs to the extended domain of the linear operator

$$Kf(x) = \int_{\mathbb{R}} k(x, y)\, f(y)\, dm(y).$$

Let us observe that if $|\phi|$ satisfies the Δ_2-condition then it suffices to consider the condition $\phi(\lambda(\cdot)) \in \tilde{D}_K$ for some $\lambda > 0$.

By the use of Theorem 3.12 we can obtain sometimes a convenient characterization of domains of nonlinear operators.

EXAMPLE 3.15. Let us consider again the nonlinear Fourier transform

$$T(f)(x) = \int_{\mathbb{R}} e^{-itx}\, \phi(f(t))\, dm(t).$$

Let H be now the space of all measurable functions M. By Theorem 3.12,

$$E_\rho = \{f \in M;\ \phi(\lambda f(\cdot)) \in \tilde{D}_{\mathscr{F}} \text{ for every } \lambda > 0\},$$

where $\mathscr{F}$ is the linear Fourier transformation

$$\mathscr{F} f(x) = \int_{\mathbb{R}} e^{-itx} f(t)\, dm(t).$$

It was proved by Szeptycki [14] (see also [10]) that $\tilde{D}_{\mathscr{F}} = l^2(L^1)$, i.e., the domain E_ρ of the operator T consists of all functions f such that

$$\sum_{-\infty}^{\infty} \left(\int_n^{n+1} |\phi(\lambda f(y))| \, dm(y) \right)^2 < \infty \qquad \text{for every} \quad \lambda > 0.$$

If $|\phi|$ satisfies the Δ_2-condition then it suffices to assume that the above inequality holds for a positive λ.

Let us observe that if we have a sequence (T_n) of nonlinear operators satisfying (3.1), (3.2), (3.3), and (3.4) with the same space H, then we can construct the respective function semimodulars ρ_n and corresponding spaces E_{ρ_n}. Defining

$$\rho(f, A) = \sum_{n=1}^{\infty} 2^{-n} \cdot \frac{\rho_n(f, A)}{1 + \rho_n(f, A)}$$

(the convention $\infty/\infty = 1$ being used) we obtain again a function semimodular (cf. [6, Sect. 8]) and corresponding space E_ρ such that every $T_n \colon E_\rho \to H$ is continuous.

We can observe that

$$L_\rho = \bigcap_{n=1}^{\infty} L_{\rho_n}, \qquad E_\rho = \bigcap_{n=1}^{\infty} E_{\rho_n}, \qquad L_\rho^0 = \bigcap_{n=1}^{\infty} L_{\rho_n}^0.$$

If (f_k) is a sequence of functions from L_ρ^0 such that $\|f_k\|_\rho \to 0$ as $k \to \infty$, then for every natural n there holds

$$\|T_n(f_k)\|_H \leqslant \rho_n(f_k) \to 0 \qquad \text{as} \quad k \to \infty.$$

The latter fact denotes simply that every operator T_n is $\|\cdot\|_\rho$-continuous at zero when regarded as an operator acting from L_ρ^0 into H. Furthermore, we can observe that even the stronger continuity holds, namely

$$\|T_n\|_H \to 0 \qquad \text{whenever} \quad \rho(f_n) \to 0 \quad \text{and} \quad f_n \in L_\rho^0.$$

An interesting question arises at this point: are T_n equicontinuous at zero in that sense? In other words, we are interested when the following is true:

$$\sup_{n \in \mathbb{N}} \|T_n(f_k)\|_H \to 0 \qquad \text{if} \quad \rho(f_k) \to 0 \quad \text{and} \quad f_k \in L_\rho^0. \tag{3.16}$$

Certainly, the above does not hold in general. Let us put, for instance, $H = \mathbb{R}$, $X = [0, 1]$, $T_n(f) = \int_X |f(x)|^n \, dm(x)$. If we put $f = 2 \cdot 1_X$ we get $f \in L_\rho^0$ while $\sup_{n \in \mathbb{N}} T_n(f) = \infty$.

To the end of the paper we will restrict our consideration to some more specified operators T_n. Let $X = \mathbb{R}$ and let m denote the Lebesgue measure in $\mathbb{R}$. Let us consider the case $H = L^\infty(\mathbb{R})$. Assume that the following estimation holds for every $n \in \mathbb{N}$ and $x \in \mathbb{R}$:

$$|T_n(f)(x)| \leqslant \int_{\mathbb{R}} a(x, y)\, \phi_n(|f(y)|)\, dm(y), \tag{3.17}$$

where a is a nonnegative measurable function and the functions ϕ_n are continuous, nondecreasing functions acting from $\mathbb{R}^+$ into itself such that $\phi_n(0) = 0$, $\phi_n(u) > 0$ for $u > 0$ and $\phi_n(u) \to \infty$ as $u \to \infty$.

We seek the necessary and sufficient conditions for the equicontinuity of the sequence of dominating operators

$$K_n(f)(x) = \int_{\mathbb{R}} a(x, y)\, \phi_n(|f(y)|)\, dm(y). \tag{3.18}$$

Let us note that the Hammerstein operators K_n satisfy conditions (3.1) through (3.4) and that the function semimodulars ρ_n induced by K_n are simply given by the formula

$$\rho_n(f, A) = \operatorname*{ess\,sup}_{x \in \mathbb{N}} \int_{\mathbb{R}} a(x, y)\, \phi_n(|f(y)|)\, dm(y),$$

$$\rho_0(f, A) = \sup_{n \in \mathbb{N}} \rho_n(f, A)$$

and

$$\rho_0(f) = \rho_0(f, \mathbb{R}) = \sup_{n \in \mathbb{N}} \rho_n(f).$$

It is well known (see, e.g., [12, p. 112]) that ρ_0 is a semimodular and $L_{\rho_0} \subset L_\rho$. Our next result is obvious in view of the definition of both semimodulars ρ and ρ_0.

PROPOSITION 3.19. *The following two statements are equivalent*:

The operators K_n are equicontinuous at zero in L^0_ρ; (3.19.a)

For every sequence of functions (f_k) from L^0_ρ there holds

$$\rho_0(f_k) \to 0 \text{ whenever } \rho(f_k) \to 0. \tag{3.19.b}$$

In the sequel we will always assume that the sequence (ϕ_k) satisfies:

Every function ϕ_k is regular, i.e., to every $\gamma_k > 0$ there exist positive numbers u'_k and α'_k such that

$$\phi_k(\alpha'_k u) \leqslant \gamma_k \phi_k(u) \qquad \text{for all} \quad u \geqslant u'_k. \tag{3.20}$$

The functions ϕ_k are equicontinuous at zero. (3.21)

The sequence (ϕ_k) is essentially increasing, i.e., to every index $n \in \mathbb{N}$ there correspond three positive numbers λ_n, β_n, v_n such that

$$\phi_n(\lambda_n u) \leqslant \beta_n \phi_k(u) \qquad \text{for all} \quad k \geqslant n, \quad u \geqslant v_n. \tag{3.22}$$

Let us give an example of an essentially increasing sequence of ϕ-functions. Define

$$\phi_k(u) = [\phi(u)]^{p_k},$$

where ϕ is an arbitrary regular ϕ-function and (p_k) is a nondecreasing sequence of numbers such that $p_k \geqslant 1$ for all $k \in \mathbb{N}$.

DEFINITION 3.23. We say that the sequence (ϕ_k) of ϕ-functions is essentially constant if there exist three positive numbers c_0, ω, u_0 and an index $i_0 \in \mathbb{N}$ such that the following estimation holds for every natural $i \geqslant i_0$ and all $u \geqslant u_0$:

$$\phi_i(\omega u) \leqslant c_0 \phi_{i_0}(u). \tag{3.23.a}$$

Remark 3.24. It was proved (cf. [12, p. 126]) that (ϕ_k) is essentially constant if and only if

There exist an index $i_0 \in \mathbb{N}$ and a number $\omega' > 0$ such that to every $u' > 0$ we can pick up a constant $c'_0 > 0$ for which $\phi_i(u) \leqslant c'_0 \phi_{i_0}(\omega' u)$ whenever $u \geqslant u'$ and $i \geqslant i_0$. (3.24.a)

Let us put

$$\mu_a(A) = \operatorname*{ess\,sup}_{x \in \mathbb{R}} \int_A a(x, y)\, dm(y), \qquad \text{where} \quad A \in \Sigma. \tag{3.25}$$

We shall assume that:

μ_a is a finite, order continuous subadditive measure; (3.26)

μ_a is equisplittable, i.e., there exists a number $\eta > 0$ such that for every sequence of numbers

$$\varepsilon_k \leqslant \eta,\ \varepsilon_k \downarrow 0,\ \varepsilon_{k+1}/\varepsilon_k \leqslant \tfrac{1}{2} \qquad (k \in \mathbb{N})$$

there exist constants

$$M \geqslant m > 0$$

and a sequence of disjoint sets $A_k \in \Sigma$ with the property

$$m\varepsilon_k \leqslant \mu_a(A_k) \leqslant M\varepsilon_k \qquad (k \in \mathbb{N}). \tag{3.27}$$

The reader is referred to [12] or [7] for the list of functions $a(x, y)$ that define equisplittable μ_a. Let us mention only that Riesz means of order $r \geqslant 1$ and Stieltjes and Abel means define equisplittable μ_a.

PROPOSITION 3.28. *Assume that ϕ_k is a regular function. If for a constant $\gamma_k > 0$ there holds*

$$\rho_k(\lambda_k f) = \operatorname*{ess\,sup}_{x \in \mathbb{R}} \int_{\mathbb{R}} a(x, y)\, \phi_k(\lambda_k |f(y)|)\, dm(y) < \infty,$$

then there exists $\delta_k > 0$ such that the function $\delta_k f$ belongs to $L^0_{\rho_k}$.

Proof. Let us fix a number $\varepsilon > 0$. Suppose $\rho_k(\lambda_k f) < \infty$ for a given $f \in M$ and $\lambda_k > 0$. Let us put

$$\gamma_k = \frac{\varepsilon}{2\rho_k(\lambda_k f)}$$

and take the numbers u', α' from the definition of regularity of ϕ_k. Then let (D_n) be an arbitrary sequence of measurable sets with $D_n \downarrow \varnothing$. Define two sequences

$$A_n = \{ y \in D_n;\, \lambda_k |f(y)| \geqslant u' \} \qquad \text{and} \qquad B_n = D_n \backslash A_n.$$

We have

$$\begin{aligned}
\rho_k(\delta_k f, D_n) &\leqslant \operatorname*{ess\,sup}_{x \in \mathbb{R}} \int_{A_n} a(x, y)\, \phi_k\left(\frac{\delta_k}{\lambda_k} \lambda_k |f(y)|\right) dm(y) \\
&\quad + \operatorname*{ess\,sup}_{x \in \mathbb{R}} \int_{B_n} a(x, y)\, \phi_k\left(\frac{\delta_k}{\lambda_k} \lambda_k |f(y)|\right) dm(y) \\
&\leqslant \operatorname*{ess\,sup}_{x \in \mathbb{R}} \int_{A_n} a(x, y)\, \phi_k(\alpha' \lambda_k |f(y)|)\, dm(y) \\
&\quad + \operatorname*{ess\,sup}_{x \in \mathbb{R}} \int_{B_n} a(x, y)\, \phi_k\left(\frac{\delta_k}{\lambda_k} u'\right) dm(y) \\
&\leqslant \gamma_k \operatorname*{ess\,sup}_{x \in \mathbb{R}} \int_{A_n} a(x, y)\, \phi_k(\lambda_k |f(y)|)\, dm(y) \\
&\quad + \phi_k\left(\frac{\delta_k}{\lambda_k} u'\right) \operatorname*{ess\,sup}_{x \in \mathbb{R}} \int_{B_n} a(x, y)\, dm(y) \\
&\leqslant \frac{\varepsilon}{2\rho_k(\lambda_k f)} \rho_k(\lambda_k f) + \phi_k\left(\frac{\delta_k}{\lambda_k} u'\right) \mu_a(B_n).
\end{aligned}$$

Since $B_n \downarrow \varnothing$, it follows by (3.26) that $\mu_a(B_n) \to 0$ as $n \to \infty$. Hence,

$$\mu_a(B_n) \leqslant \frac{\varepsilon}{2}\left\{\phi_k\left(\frac{\delta_k}{\lambda_k} u'\right)\right\}^{-1} \qquad \text{for} \quad n \text{ sufficiently large}$$

and finally

$$\rho_k(\delta_k f, D_n) < \varepsilon \qquad \text{for} \quad n \text{ sufficiently large,}$$

which means, in view of arbitrariness of $D_n \downarrow \varnothing$, that the function $\delta_k f$ is a member of the set $L^0_{\rho_k}$.

The next result gives the characterization of equicontinuity of (K_n).

THEOREM 3.29. *The following conditions are equivalent*:

$$\text{The operators } K_n \text{ are equicontinuous at zero in } L^0_\rho; \tag{3.29.a}$$

$$\text{The sequence of functions } (\phi_k) \text{ is essentially constant.} \tag{3.29.b}$$

Proof. (3.29.b) $\Rightarrow$ (3.29.a). In view of Proposition 3.19 we have to show that if $f_k \in L^0_\rho$ and $\rho(f_k) \to 0$ then $\rho_o(f_k) \to 0$. Fix a sequence $(f_k) \subset L^0_\rho$ as an arbitrary number $\varepsilon > 0$. Since ϕ_i are equicontinuous at zero and

$$\operatorname*{ess\,sup}_{x \in \mathbb{R}} \int_{\mathbb{R}} a(x, y)\, dm(y) = c < \infty,$$

we may find $u' > 0$ such that

$$\phi_i(u') < \varepsilon/2c \qquad \text{for all indices} \quad \in \mathbb{N}.$$

Let us fix an index i_0 and pick up a positive number ω' existing by (3.24.a). We may, therefore, find $c_0' > 0$ corresponding to u' and, hence, obtain

$$\phi_i(u) \leqslant c_0' \phi_{i_0}(\omega' u) \qquad \text{for} \quad u \geqslant u' \quad \text{and} \quad i \geqslant i_0.$$

Let $k_0 \in \mathbb{N}$ be so large that for every $k \geqslant k_0$ there hold

$$\rho_i(f_k) < \varepsilon \qquad \text{for} \quad i = 1, 2, ..., i_0 - 1$$

and

$$\rho_{i_0}(\omega' f_k) \leqslant \varepsilon/2c_0'.$$

Let us put

$$A_k = \{y \in \mathbb{R};\, |f_k(y)| \geqslant u'\} \qquad \text{and} \qquad B_k = \mathbb{R} \backslash A_k.$$

For $i \geqslant i_0$ and $k \geqslant k_0$ we obtain

$$\begin{aligned}\rho_i(f_k) &\leqslant \operatorname*{ess\,sup}_{x \in \mathbb{R}} \int_{A_k} a(x, y)\, \phi_i(|f_k(y)|)\, dm(y)\\ &\quad + \operatorname*{ess\,sup}_{x \in \mathbb{R}} \int_{B_k} a(x, y)\, \phi_i(|f_k(y)|)\, dm(y)\\ &\leqslant c_0' \operatorname*{ess\,sup}_{x \in \mathbb{R}} \int_{A_k} a(x, y)\, \phi_{i_0}(\omega' |f_k(y)|)\, dm(y)\\ &\quad + \phi_i(u') \operatorname*{ess\,sup}_{x \in \mathbb{R}} \int_{B_k} a(x, y)\, dm(y)\\ &\leqslant c_0' \rho_{i_0}(\omega' f_k) + \phi_i(u') c < \varepsilon.\end{aligned}$$

Finally, for all $i \in \mathbb{N}$ there holds

$$\rho_i(f_k) < \varepsilon \qquad \text{whenever} \quad k \geqslant k_0.$$

Hence,

$$\rho_0(f_k) = \sup_{i \in \mathbb{N}} \rho_i(f_k) < \varepsilon \qquad \text{for} \quad k \geqslant k_0,$$

i.e.,

$$\rho_0(f_k) \to 0 \qquad \text{as} \quad k \to \infty.$$

(3.29.a) $\Rightarrow$ (3.29.b). Assume to the contrary that (ϕ_k) is not essentially constant. It follows from Theorem 17.9 in [12] that there exists function f such that $f \in L_\rho \backslash L_{\rho_0}$. Since f does not belong to L_{ρ_0}, there exist $\varepsilon > 0$ and a sequence $0 \leqslant \delta_k \downarrow 0$ such that

$$\rho_0(\delta_k f) > \varepsilon > 0.$$

On the other hand, f does belong to L_ρ and, therefore, there exists $k_0 \in \mathbb{N}$ such that for every $k \geqslant k_0$ there holds

$$\rho(\delta_k f) < \infty.$$

In view of Proposition 3.28, we conclude then that $\delta_k f$ belongs to L_ρ^0 for k sufficiently large and, since $\rho(\delta_k f) \to 0$, we obtain the contradiction with the condition (3.19.b), which is equivalent to (3.29.a). The contradiction completes the proof.

Let us observe that if (ϕ_k) is an essentially constant sequence of regular ϕ-functions there exists then $i_0 \in \mathbb{N}$ such that ϕ_i is equivalent to ϕ_{i_0} for every $i \geqslant i_0$, which implies the equivalence of modulars ρ_i and ρ_{i_0}. The latter denotes simply that for every sequence $(f_k) \subset L_\rho^0$ and every $i \geqslant i_0$ the following statement is true:

$$\|K_i(f_k)\|_H \to 0 \qquad \text{if and only if} \quad \|K_{i_0}(f_k)\|_H \to 0.$$

Therefore, the meaning of Theorem 3.29 is as follows: the construction of the semimodular ρ guarantees certainly the equicontinuity of every finite number of operators K_n but one should not expect any interesting result for infinite sequence of operators.

ACKNOWLEGDMENTS

The author would like to express his sincere gratitude to Professors I. Labuda, W. A. J. Luxemburg, and P. Szeptycki for their inspiring remarks about the subject of this paper.

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